Eur. Phys. J. C 25, 333–338 (2002)
Digital Object Identifier (DOI) 10.1007/s10052-002-0984-0

THE EUROPEAN
PHYSICAL JOURNAL C

Rabi oscillations in gravitational fields: Exact solution via path integral

M. Aouachria, L. Chetouani

Département de Physique, Faculté des Sciences, Université Mentouri, 25000 Constantine, Algeria

Received: 15 March 2002 / Revised version: 15 April 2002 /
Published online: 12 July 2002 –

Abstract. The movement of a two-level atom interacting with an electromagnetic wave and subject to gravitation is studied using the path-integral formalism. The propagator is first of all written in the standard form $\int \mathcal{D}(\text{path}) \exp(\mathrm{i}/\hbar) S(\text{path})$ by replacing the spin by two fermionic oscillators; then it is determined exactly due to the auxiliary equation which has a cylindric parabolic function as a solution.

1 Introduction

Up to now, a whole class of potentials have been treated successfully within the path-integral formalism, thanks to the use of certain transformations [1]. However, it is known that the most relativistic interactions are those where spin is taken into account, which is a very useful and very important notion in physics. In practice, explicit calculations of propagators for such interactions by the path-integral formalism are very scarce [2].

For this reason we are devoted to this type of interaction; we consider this problem, as treated recently according to the usual quantum mechanics [3]. It occurs for an atom which has two levels and which interacts with a electromagnetic wave polarized circularly. Its dynamics is described by the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} - m\boldsymbol{g}\boldsymbol{r} + \frac{1}{2}\left(E_a + E_b\right)\boldsymbol{I} + \frac{1}{2}\hbar\omega_{ba}\boldsymbol{\sigma}_z - \hbar\Omega_{ba}\left(\mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}\boldsymbol{\sigma}_+ + \mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}\boldsymbol{\sigma}_-\right), \quad (1)$$

where the first term represents the kinetic energy, the second term represents the gravitation energy, the fourth term represents the interaction energy of the internal motion (spin) with a two-level atom whose energy levels are E_b and E_a, $\omega_{ba} = (1/\hbar)\left(E_b - E_a\right)$, and the fifth term is a scalar electric dipole interaction.

The Pauli matrices are the following:

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2)$$

As in any experiment which is done in the laboratory, the influence of the earth's acceleration in dynamics (the second term) is taken in account. Thus, by calculating the probability of a Rabi transition and the transfer from the population [3], the influence of the gravitation explicitly was underlined.

In this work we propose to give an alternative solution to the same problem by the path-integral formalism. For that we show

(1) that such a problem is reduced to the usual form $\int \mathcal{D}(\text{path}) \exp(\mathrm{i}/\hbar) S(\text{path})$ of Feynman which utilizes in addition the path grassmanian;
(2) that then the solution requires only the knowledge of the equation of the movement of the atom, and some transformations.

2 Path-integral formulation

There are many ways for introducing the spin in the path-integral formalism [4,5]. The easiest way consists of replacing $\boldsymbol{\sigma}$ by two fermionic oscillators (u, d) [6]. We have

$$\begin{cases} \boldsymbol{\sigma}_z \longrightarrow u^\dagger u - d^\dagger d, \\ \boldsymbol{\sigma}_+ \longrightarrow u^\dagger d, \\ \boldsymbol{\sigma}_- \longrightarrow u d^\dagger, \end{cases} \quad (3)$$

or in condensed notation

$$\boldsymbol{\sigma} \longrightarrow \left(u^\dagger, d^\dagger\right) \boldsymbol{\sigma} \begin{pmatrix} u \\ d \end{pmatrix}. \quad (4)$$

Obviously, the operators u, d anticommute between themselves. This fact allows us to write the Hamiltonian in the following oscillator form:

$$H = \frac{\boldsymbol{p}^2}{2m} - m\boldsymbol{g}\boldsymbol{r} + \frac{1}{2}\left(E_a + E_b\right)\boldsymbol{I} + \frac{1}{2}\hbar\omega_{ba}(u^\dagger u - d^\dagger d) - \hbar\Omega_{ba}\left(\mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}u^\dagger d + \mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}d^\dagger u\right). \quad (5)$$

Now, we move to the path-integral formulation. We designate by $\boldsymbol{r}$ the real variable describing the atom position, and by (α, β) the Grassmann variables to describe the dynamics of the spin. These variables have the following characteristic features:

$$\begin{cases} u \mid \alpha, \beta\rangle = \alpha \mid \alpha, \beta\rangle \quad \text{and} \langle \alpha, \beta \mid u^\dagger = \langle \alpha, \beta \mid \overline{\alpha}, \\ d \mid \alpha, \beta\rangle = \beta \mid \alpha, \beta\rangle \quad \text{and} \langle \alpha, \beta \mid d^\dagger = \langle \alpha, \beta \mid \overline{\beta}, \\ \langle \alpha, \beta \mid \alpha', \beta'\rangle = \mathrm{e}^{\overline{\alpha}\alpha' + \overline{\beta}\beta'}. \end{cases} \tag{6}$$

Our purpose is to calculate the matrix element of the following evolution operator:

$$\boldsymbol{U}(T;0) = T_{\mathrm{D}} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T H \mathrm{d}t\right), \tag{7}$$

namely

$$\boldsymbol{K}(f,i;T) = \langle \boldsymbol{r}_f, \alpha_f, \beta_f \mid \hat{\boldsymbol{U}}(T) \mid \boldsymbol{r}_i, \alpha_i, \beta_i\rangle, \tag{8}$$

where T_{D} is the time ordered Dyson symbol.

Let us subdivide the time interval $[0,T]$ into $N+1$ intervals of length ϵ, using the Trotter formula and inserting N times the completeness relations in order to eliminate the operators that appear in the exponential. We get

$$\int \mid \boldsymbol{r}\rangle\langle \boldsymbol{r} \mid \mathrm{d}^3 r = 1, \tag{9}$$
$$\int \mathrm{d}\overline{\alpha}\mathrm{d}\alpha' \mathrm{d}\overline{\beta}\mathrm{d}\beta' \mathrm{e}^{-\overline{\alpha}\alpha' - \overline{\beta}\beta'} \mid \alpha', \beta'\rangle\langle \alpha, \beta \mid = 1.$$

It is easy to show that the propagator takes the following discretized form:

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \lim_{N\longrightarrow\infty} \left(\frac{m}{2\pi\hbar \mathrm{i}\epsilon}\right)^{3N/2} \int \prod_{n=1}^{N} \mathrm{d}^3 r_n \prod_{n=1}^{N+1} \\ & \times \exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{m}{2\epsilon}(\boldsymbol{\Delta r}_n)^2 + \epsilon m \boldsymbol{g}\boldsymbol{r}_n + \frac{\epsilon}{2}(E_a + E_b)\right]\right\} \\ & \times \lim_{N\longrightarrow\infty} \int \prod_{n=1}^{N} \mathrm{d}\bar{\alpha}_n \mathrm{d}\alpha_n \mathrm{d}\bar{\beta}_n \mathrm{d}\beta_n \mathrm{e}^{-\bar{\alpha}_n\alpha_n - \bar{\beta}_n\beta_n} \\ & \times \prod_{n=1}^{N+1} \exp\left\{\left[\left(1 - \mathrm{i}\epsilon\frac{\omega_{ba}}{2}\right)\bar{\alpha}\alpha_{n-1}\right.\right. \\ & + \left(1 + \mathrm{i}\epsilon\frac{\omega_{ba}}{2}\right)\overline{\beta}_n\beta_{n-1} + \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{\mathrm{i}(\omega t_n - \boldsymbol{k}\boldsymbol{r}_{n-1} + \phi)}\overline{\beta}_n\alpha_{n-1} \\ & \left.\left. + \ \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{-\mathrm{i}(\omega t_n - \boldsymbol{k}\boldsymbol{r}_n + \phi)}\bar{\alpha}_n\beta_{n-1}\right]\right\}, \end{aligned} \tag{10}$$

or, in continuous form,

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \int \mathcal{D}^3 r \mathcal{D}\overline{\alpha}\mathcal{D}\alpha\mathcal{D}\overline{\beta}\mathcal{D}\beta \\ & \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\left[\frac{m}{2}\dot{\boldsymbol{r}}^2 + \frac{\mathrm{i}\hbar}{2}(\overline{\alpha}\,\dot{\alpha} + \overline{\beta}\,\dot{\beta} - \dot{\overline{\alpha}}\,\alpha - \dot{\overline{\beta}}\,\beta)\right.\right. \\ & + m\boldsymbol{g}\boldsymbol{r} + \frac{1}{2}(E_a + E_b) - \mathrm{i}\hbar\frac{\omega_{ba}}{2}(\overline{\alpha}\alpha - \overline{\beta}\beta) \\ & \left.\left. + \mathrm{i}\hbar\Omega_{ba}(\mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}\overline{\beta}\alpha + \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\boldsymbol{r} + \phi)}\overline{\alpha}\beta)\right]\right\} \\ = & \int \mathcal{D}(\text{path}) \exp\frac{\mathrm{i}}{\hbar}S(\text{path}). \end{aligned} \tag{11}$$

3 The propagator calculation

In order to remove the terms in $\exp(\pm\mathrm{i}\boldsymbol{k}\boldsymbol{r})$, let us introduce the first following transformation:

$$\begin{cases} \alpha \longmapsto \eta, \\ \alpha_n = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}_n}\eta_n, \end{cases} \quad \text{and} \quad \begin{cases} \bar{\alpha} \longmapsto \overline{\eta}, \\ \bar{\alpha}_n = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_n}\overline{\eta}_n. \end{cases} \tag{12}$$

Now the product $\bar{\alpha}_n\alpha_{n-1}$ gives

$$\begin{aligned} \bar{\alpha}_n\alpha_{n-1} &= \overline{\eta}_n\eta_{n-1}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\Delta r}_n} \\ &= \overline{\eta}_n\eta_{n-1}\left(1 - \mathrm{i}\boldsymbol{k}\boldsymbol{\Delta r}_n - \frac{1}{2}(\boldsymbol{k}\boldsymbol{\Delta r}_n)^2 + \mathcal{O}\left((\boldsymbol{\Delta r})^3\right)\right), \end{aligned} \tag{13}$$

where $\boldsymbol{\Delta r}_n = \boldsymbol{r}_n - \boldsymbol{r}_{n-1}$. The other terms of order greater than 2 and present in the infinitesimal action have been neglected since they did not contribute to the path integral. The third term in the development gives

$$\begin{aligned} (\boldsymbol{k}\boldsymbol{\Delta r}_n)^2 = & k_x^2(\Delta x_n)^2 + k_y^2(\Delta y_n)^2 + k_z^2(\Delta z_n)^2 \\ & + 2(k_x k_y \Delta x_n \Delta y_n + k_x k_z \Delta x_n \Delta z_n + k_y k_z \Delta y_n \Delta z_n). \end{aligned} \tag{14}$$

All these corrections can be estimated by the following equations:

$$\begin{aligned} \langle(\Delta x_n)^2\rangle &= \langle(\Delta y_n)^2\rangle \\ &= \langle(\Delta z_n)^2\rangle \\ &= \sqrt{\frac{m}{2\pi\mathrm{i}\epsilon}}\int \mathrm{d}(\Delta y_n)(\Delta y_n)^2\mathrm{e}^{(\mathrm{i}m)/(2\epsilon)(\Delta y_n)^2} \\ &= \mathrm{i}\frac{\hbar\epsilon}{m}, \end{aligned} \tag{15}$$

$$\langle\Delta x_n \Delta y_n\rangle = \langle\Delta x_n \Delta z_n\rangle = \langle\Delta y_n \Delta z_n\rangle = 0. \tag{16}$$

The contribution of these corrections consists of the introduction of an effective potential in the action as follows:

$$\left\langle(\boldsymbol{k}\boldsymbol{\Delta r}_n)^2\right\rangle = \frac{\mathrm{i}\epsilon\hbar\boldsymbol{k}^2}{2m}. \tag{17}$$

Then, taking into account that the Grasmmann variable verifies $\eta^2 = 0$, the propagator (10) takes the following discretized form:

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \lim_{N\longrightarrow\infty} \int \prod_{n=1}^{N} \mathrm{d}\overline{\eta}_n \mathrm{d}\eta_n \mathrm{d}\bar{\beta}_n \mathrm{d}\beta_n \mathrm{e}^{-\overline{\eta}_n\eta_{n-1} - \bar{\beta}_n\beta_n} \\ & \times \prod_{n=1}^{N+1} \exp\left\{\left[\left(1 - \mathrm{i}\epsilon\frac{\omega_{ba}}{2}\right)\overline{\eta}_n\eta_{n-1} + \left(1 + \mathrm{i}\epsilon\frac{\omega_{ba}}{2}\right)\bar{\beta}_n\beta_{n-1}\right.\right. \\ & - \mathrm{i}\frac{\epsilon\boldsymbol{k}^2}{2m}\overline{\eta}_n\eta_{n-1} + \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{\mathrm{i}(\omega t_n + \phi)}\bar{\beta}_n\eta_{n-1} \end{aligned}$$

$$+\ \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{-\mathrm{i}(\omega t_n+\phi)}\bar{\eta}_n\beta_{n-1}\Big]\Big\}\left(\frac{m}{2\pi\hbar\mathrm{i}\epsilon}\right)^{3N/2}$$
$$\times\int\prod_{n=1}^{N}\mathrm{d}^3r\prod_{n=1}^{N+1}\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{m}{2\epsilon}\left(\Delta\boldsymbol{r}-\epsilon\frac{\hbar\boldsymbol{k}}{m}\bar{\eta}_n\eta_{n-1}\right)^2\right.\right.$$
$$\left.\left.+\ \epsilon m\boldsymbol{g}\boldsymbol{r}+\frac{\epsilon}{2}\left(E_a+E_b\right)\right]\right\}. \tag{18}$$

It is better to linearize the kinetic energy using the phase space, and then integrate over $\boldsymbol{r}$. So

$$\boldsymbol{K}(f,i;T)=\int\mathcal{D}^3p\mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\boldsymbol{r}\,|_0^T}\,(2\pi\hbar)^3\,\delta\left(\dot{\boldsymbol{p}}-m\boldsymbol{g}\right)$$
$$\times\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^T\mathrm{d}t\left[\frac{\boldsymbol{p}^2}{2m}+\frac{\boldsymbol{p}\boldsymbol{k}}{m}\bar{\eta}\eta\right]\right\}$$
$$\times\ \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T}\int\mathcal{D}\bar{\eta}\mathcal{D}\eta\mathcal{D}\bar{\beta}\mathcal{D}\beta$$
$$\times\exp\left\{\int_0^T\left[-\frac{1}{2}(\bar{\eta}\,\dot{\eta}+\bar{\beta}\,\dot{\beta}-\dot{\bar{\eta}}\,\eta-\dot{\bar{\beta}}\,\beta)\right.\right.$$
$$-\ \mathrm{i}\frac{\omega_{ba}}{2}(\bar{\eta}\eta-\bar{\beta}\beta) \tag{19}$$
$$\left.\left.+\ \mathrm{i}\Omega_{ba}(\mathrm{e}^{\mathrm{i}(\omega t+\phi)}\bar{\beta}\eta+\mathrm{e}^{-\mathrm{i}(\omega t+\phi)}\bar{\eta}\beta)-\mathrm{i}\frac{\hbar\boldsymbol{k}^2}{2m}\bar{\eta}\eta\right]\right\}.$$

Clearly, the argument of the Dirac function $\delta\left(\dot{\boldsymbol{p}}-m\boldsymbol{g}\right)$ means that the particle is only subject to the action of gravitation. The atom momentum is

$$\boldsymbol{p}=m\boldsymbol{g}t+\boldsymbol{p}_0\quad\text{where}\quad(\boldsymbol{p}_0=\text{constant}). \tag{20}$$

The contribution of the time-linear function in the computation of the propagator has the following result:

$$\boldsymbol{K}(f,i;T)=\int\frac{\mathrm{d}^3p_0}{(2\pi\hbar)^3}\mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T}$$
$$\times\ \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2/2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)}\mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T}$$
$$\times\lim_{N\longrightarrow\infty}\int\prod_{n=1}^{N}\mathrm{d}\bar{\eta}_nd\eta_nd\bar{\beta}_n\mathrm{d}\beta_n\mathrm{e}^{-\bar{\eta}_n\eta_n-\bar{\beta}_n\beta_n}$$
$$\times\prod_{n=1}^{n=N+1}\exp\left\{\left(\bar{\eta}_n,\bar{\beta}_n\right)\right.$$
$$\begin{pmatrix}1-\mathrm{i}\epsilon\left(\frac{\omega_{ba}}{2}+\frac{\hbar\boldsymbol{k}^2}{2m}+\frac{\boldsymbol{k}}{m\hbar}\left(m\boldsymbol{g}t_n+\boldsymbol{p}_0\right)\right) & \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{-\mathrm{i}(\omega t_n+\phi)}\\ \mathrm{i}\epsilon\Omega_{ba}\mathrm{e}^{\mathrm{i}(\omega t_n+\phi)} & 1+\mathrm{i}\frac{\omega_{ba}}{2}\end{pmatrix}$$
$$\left.\times\begin{pmatrix}\eta_{n-1}\\ \beta_{n-1}\end{pmatrix}\right\}. \tag{21}$$

At this level, let us deal with the integration over the Grassmann variables. We shall remove the terms in $\mathrm{e}^{\pm\mathrm{i}(\omega t+\varphi)}$ by introducing the following change:

$$\begin{cases}(\eta,\beta)\longrightarrow(\psi,\phi)\,,\\ \begin{pmatrix}\eta\\ \beta\end{pmatrix}=\mathrm{e}^{-(\mathrm{i}/2)(\omega t+\phi)\boldsymbol{\sigma}_z}\\ \qquad\times\mathrm{e}^{-(\mathrm{i}/\hbar)\left[(\hbar^2\boldsymbol{k}^2)/(4m)+(\boldsymbol{k}/2m)\left((1/2)m\boldsymbol{g}t^2+\boldsymbol{p}_0t\right)\right]}\begin{pmatrix}\psi\\ \phi\end{pmatrix},\end{cases} \tag{22}$$

and

$$\begin{cases}\left(\bar{\eta},\bar{\beta}\right)\longrightarrow\left(\bar{\psi},\bar{\phi}\right),\\ \left(\bar{\eta},\bar{\beta}\right)=\mathrm{e}^{(\mathrm{i}/\hbar)\left[(\hbar^2\boldsymbol{k}^2)/(4m)+(\boldsymbol{k}/2m)\left((1/2)m\boldsymbol{g}t^2+\boldsymbol{p}_0t\right)\right]}\\ \quad\times\mathrm{e}^{(\mathrm{i}/2)(\omega t+\phi)\boldsymbol{\sigma}_z}\left(\bar{\psi},\bar{\phi}\right),\end{cases} \tag{23}$$

So the expression of the propagator (20) becomes

$$\boldsymbol{K}(f,i;T)=\int\frac{\mathrm{d}^3p_0}{(2\pi\hbar)^3}\mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T}$$
$$\times\ \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)}$$
$$\times\ \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T}$$
$$\times\lim_{N\longrightarrow\infty}\int\prod_{n=1}^{n=N}\mathrm{d}\bar{\psi}_n\mathrm{d}\psi_n\mathrm{d}\bar{\phi}_n\mathrm{d}\phi_n\mathrm{e}^{-\bar{\psi}_n\psi_n-\bar{\phi}_n\phi_n} \tag{24}$$
$$\times\prod_{n=1}^{N+1}\exp\left\{\left(\bar{\psi}_n,\bar{\phi}_n\right)\begin{pmatrix}1-\tau\frac{\lambda_n}{2} & \tau\sqrt{\nu}\\ \tau\sqrt{\nu} & 1+\tau\frac{\lambda_n}{2}\end{pmatrix}\begin{pmatrix}\psi_{n-1}\\ \phi_{n-1}\end{pmatrix}\right\},$$

where the new parameter has been introduced

$$\begin{cases}\lambda=\mathrm{e}^{\mathrm{i}\pi/4}\left[\frac{2\Omega_{ba}y(\boldsymbol{p}_0)}{\sqrt{\boldsymbol{k}\boldsymbol{g}}}+\sqrt{\boldsymbol{k}\boldsymbol{g}}t\right], & \epsilon=\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\boldsymbol{k}\boldsymbol{g}}}\tau,\\ & \nu=\mathrm{i}\frac{\Omega_{ba}^2}{\boldsymbol{k}\boldsymbol{g}},\\ y(\boldsymbol{p}_0)=\frac{1}{2\Omega_{ba}}\left[\frac{\hbar\boldsymbol{k}}{2m}(2\boldsymbol{p}_0+\boldsymbol{k})-\Delta\right], & \Delta=\omega-\omega_{ba}.\end{cases} \tag{25}$$

The next step consists of taking the diagonal form for the action in order to be able to integrate.

Thus, we set a unit transformation over the Grassmann variables

$$\begin{cases}(\psi,\phi)\longrightarrow(\gamma,\delta)\,,\\ \begin{pmatrix}\psi\\ \phi\end{pmatrix}=U(\lambda)\begin{pmatrix}\gamma\\ \delta\end{pmatrix}=\begin{pmatrix}A(\lambda) & -B^*(\lambda)\\ B(\lambda) & A^*(\lambda)\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix},\\ (\bar{\psi},\bar{\phi})\longrightarrow(\bar{\gamma},\bar{\delta}),\\ (\bar{\psi},\bar{\phi})=U^\dagger(\lambda)(\bar{\gamma},\bar{\delta})=\begin{pmatrix}A^*(\lambda) & B^*(\lambda)\\ -B(\lambda) & A(\lambda)\end{pmatrix}(\bar{\gamma},\bar{\delta}),\end{cases} \tag{26}$$

with

$$\begin{gathered}U(\lambda)U^\dagger(\lambda)=U^\dagger(\lambda)U(\lambda)=1,\\ \det U(\lambda)=1,\end{gathered} \tag{27}$$

and the initial conditions

$$A(\lambda_0)=1,\quad B(\lambda_0)=0\quad\text{for }\lambda_0=\mathrm{e}^{\mathrm{i}\pi/4}\frac{2\Omega_{ba}y(\boldsymbol{p}_0)}{\sqrt{\boldsymbol{k}\boldsymbol{g}}}. \tag{28}$$

By means of a simple calculation including the following development

$$U(\lambda_{n-1}) = U(\lambda_n) - \tau \frac{\mathrm{d}U}{\mathrm{d}\lambda}(\lambda_n), \tag{29}$$

$$U^\dagger(\lambda_n) U(\lambda_{n-1}) = \boldsymbol{I} - \tau U^\dagger(\lambda_n) \frac{\mathrm{d}U}{\mathrm{d}\lambda}(\lambda_n), \tag{30}$$

we obtain

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \lim_{N\to\infty} \prod_{n=1}^{N} \int \mathrm{d}\bar{Z}_n \mathrm{d}Z_n \mathrm{e}^{-\bar{Z}_n Z_n} \\ & \times \prod_{n=1}^{N+1} \exp\left\{\bar{Z}_n Z_{n-1} + \mathrm{i}\tau \bar{Z}_n Q(n) Z_{n-1}\right\}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} Q(\lambda_n) = & \, \mathrm{i}U^\dagger(\lambda_n)\frac{\mathrm{d}U}{\mathrm{d}\lambda}(\lambda_n) \\ & + U^\dagger(\lambda_n) \begin{pmatrix} \mathrm{i}\frac{\lambda_n}{2} & -\mathrm{i}\sqrt{\nu} \\ -\mathrm{i}\sqrt{\nu} & -\mathrm{i}\frac{\lambda_n}{2} \end{pmatrix} U(\lambda_n), \end{aligned} \tag{32}$$

and

$$Z_n = \begin{pmatrix} \gamma_n \\ \delta_n \end{pmatrix}; \quad \bar{Z}_n = (\bar{\gamma}_n, \bar{\delta}_n). \tag{33}$$

Now, we determine the unit transformation by fixing the diagonal form for the action, which leads us to the following condition:

$$Q(\lambda_n) = 0. \tag{34}$$

To be able to integrate, we have to write the grassmannian part of the expression in an appropriate form. In fact, $\boldsymbol{K}$ is written again in the following form:

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \\ & \times \int \mathrm{d}\boldsymbol{\xi}^\dagger \mathrm{d}\boldsymbol{\xi} \exp\left[-\boldsymbol{\xi}^\dagger\boldsymbol{\xi} + \boldsymbol{V}^\dagger\boldsymbol{\xi} + \boldsymbol{\xi}^\dagger\boldsymbol{W}\right], \end{aligned} \tag{35}$$

where

$$\boldsymbol{V}^\dagger = \left(0, ..., \bar{Z}_{N+1}\right), \quad \boldsymbol{\xi} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_N \end{pmatrix}, \quad \boldsymbol{W} = \begin{pmatrix} Z_0 \\ \vdots \\ 0 \end{pmatrix}. \tag{36}$$

Now, we absorb the linear terms in $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^\dagger$ thanks to the shift

$$\boldsymbol{\xi} \to \boldsymbol{\xi} + \boldsymbol{W}, \tag{37}$$

$$\boldsymbol{\xi}^\dagger \to \boldsymbol{\xi}^\dagger + \boldsymbol{V}^\dagger,$$

and we integrate over the Grassmann variables.

Our propagator relative to the atom subject to gravity and interacting with the electromagnetic wave is finally written as follows:

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(i/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \exp\left[\bar{Z}_f Z_i\right]. \end{aligned} \tag{38}$$

In terms of the old variables it becomes

$$\begin{aligned} \boldsymbol{K}(f,i;T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \\ & \times \exp\left\{\left(\bar{\alpha}_f, \bar{\beta}_f\right) \boldsymbol{S}(T) \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}\right\}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \boldsymbol{S}(T) = & \, \mathrm{e}^{-(\mathrm{i}/\hbar)\left[(\hbar^2\boldsymbol{k}^2)/(4m)T+(\boldsymbol{k}/2m)\left((1/2)m\boldsymbol{g}T^2+\boldsymbol{p}_0T\right)\right]} \\ & \times \begin{pmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}_f} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{-(\mathrm{i}/2)(\omega T+\phi)\boldsymbol{\sigma}_z} \\ & \times \begin{pmatrix} A(T) & -B^*(T) \\ B(T) & A^*(T) \end{pmatrix} \mathrm{e}^{(\mathrm{i}/2)\phi\boldsymbol{\sigma}_z} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_i} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{40}$$

a 2×2 matrix.

Let us now turn to the calculation of this propagator, (37), between the spin states. We just evaluate the matrix $K_{\uparrow\uparrow}(\boldsymbol{r}_f, \boldsymbol{r}_i; T)$ only, and all the other matrices can be deduced following the same method. In fact, the propagator on the spin eigenstates is given by

$$\begin{aligned} \boldsymbol{K}_{\uparrow\uparrow}(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = & \int \mathrm{d}\bar{\alpha}_f \mathrm{d}\alpha_f \mathrm{d}\bar{\beta}_f \mathrm{d}\beta_f \mathrm{d}\bar{\alpha}_i \mathrm{d}\alpha_i \mathrm{d}\bar{\beta}_i \mathrm{d}\beta_i \\ & \times \mathrm{e}^{-\bar{\alpha}_f\alpha_f - \bar{\beta}_f\beta_f} \mathrm{e}^{-\bar{\alpha}_i\alpha_i - \bar{\beta}_i\beta_i} \langle\uparrow|\, \alpha_f, \beta_f\rangle \boldsymbol{K}(f,i,t) \langle\alpha_i, \beta_i\, |\uparrow\rangle. \end{aligned} \tag{41}$$

Thanks to the features [8]

$$\langle\uparrow|\, \alpha_f, \beta_f\rangle = \alpha_f, \qquad \langle\alpha_i, \beta_i\, |\uparrow\rangle = \bar{\alpha}_i, \tag{42}$$

$$\alpha_f\bar{\alpha}_i = \mathrm{e}^{-\bar{\alpha}_i\alpha_f} - 1, \tag{43}$$

(39) takes the following form:

$$\begin{aligned} K_{\uparrow\uparrow}(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \\ & \times \int \mathrm{d}\nu^\dagger \mathrm{d}\nu \left[\exp \nu^\dagger M^{'} \nu - \exp \nu^\dagger M \nu\right], \end{aligned} \tag{44}$$

where the matrices M' and M are, respectively,

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ \boldsymbol{S}_{11} & -1 & \boldsymbol{S}_{11} & 0 \\ 0 & 0 & -1 & 0 \\ \boldsymbol{S}_{21} & 0 & \boldsymbol{S}_{22} & -1 \end{pmatrix}$$

and

$$M' = \begin{pmatrix} -1 & -1 & 0 & 0 \\ \boldsymbol{S}_{11} & -1 & \boldsymbol{S}_{11} & 0 \\ 0 & 0 & -1 & 0 \\ \boldsymbol{S}_{21} & 0 & \boldsymbol{S}_{22} & -1 \end{pmatrix}, \tag{45}$$

and $\boldsymbol{S}_{nm}$ are the elements of the matrix $\boldsymbol{S}(T)$ and

$$\nu = \begin{pmatrix} \alpha_i \\ \alpha_f \\ \beta_i \\ \beta_f \end{pmatrix}, \quad \nu^{\dagger} = \begin{pmatrix} \overline{\alpha}_i & \overline{\alpha}_f & \overline{\beta}_i & \overline{\beta}_f \end{pmatrix} \tag{46}$$

are the vectors gathering the old Grassmann variables.

The integration over the Grassmann variables is thus simple:

$$\begin{aligned} K_{\uparrow\uparrow}(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \left[\det M' - \det M\right]. \end{aligned} \tag{47}$$

As $\det M' = 1 + \boldsymbol{S}_{11}$ and $\det M = 1$, the propagator following the states of the up–up spin is finally

$$\begin{aligned} K_{\uparrow\uparrow}(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \boldsymbol{S}_{11}(T). \end{aligned} \tag{48}$$

Normally, if we repeat the calculations by considering all the initial and final states of the spin, the propagator will take the following matrix form:

$$\begin{aligned} K(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = & \int \frac{\mathrm{d}^3 p_0}{(2\pi\hbar)^3} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}t+\boldsymbol{p}_0)\boldsymbol{r}|_0^T} \\ & \times \mathrm{e}^{-(\mathrm{i}/\hbar)\left((\boldsymbol{p}_0^2)/(2m)T+(1/2)\boldsymbol{g}\boldsymbol{p}_0T^2+(1/6)m\boldsymbol{g}^2T^3\right)} \\ & \times \mathrm{e}^{(\mathrm{i}/2\hbar)(E_a+E_b)T} \boldsymbol{S}(T). \end{aligned} \tag{49}$$

At this level, we notice that the unit matrix in the matrix $\boldsymbol{S}(T)$ is developed as follows:

$$\boldsymbol{I} = \eta_+ \boldsymbol{\eta}_+^+ + \boldsymbol{\eta}_- \boldsymbol{\eta}_-^+, \tag{50}$$

where

$$\boldsymbol{\eta}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\eta}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{51}$$

are the proper states of the spin.

By a comparison with the usual spectral decomposition

$$\boldsymbol{K}(\boldsymbol{r}_f, \boldsymbol{r}_i; T) = \sum_{\sigma=\pm} \int \mathrm{d}^3 p_0 \Psi_{\boldsymbol{p}_0,\sigma}(\boldsymbol{r}_f, T) \Psi^{\dagger}_{\boldsymbol{p}_0,\sigma}(\boldsymbol{r}_i, 0), \tag{52}$$

the wave functions can be deduced. They become equal to

$$\begin{aligned} \Psi_{\boldsymbol{p}_0,\pm}(\boldsymbol{r}, T) = & \frac{1}{(2\pi\hbar)^{3/2}} \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}T+\boldsymbol{p}_0)\boldsymbol{r}} \mathrm{e}^{-(\mathrm{i}/\hbar)E_{\mathrm{g}}(T)} \\ & \times \begin{pmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{-(\mathrm{i}/2)(\omega T+\phi)\sigma_z} \\ & \times \begin{pmatrix} A(T) & -B^*(T) \\ B(T) & A^*(T) \end{pmatrix} \boldsymbol{\eta}_{\pm}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} E_{\mathrm{g}}(T) = & \frac{\hbar^2 \boldsymbol{p}_0^2}{2m} T + \frac{1}{2}\hbar \left(\boldsymbol{p}_0 + \frac{1}{2}\boldsymbol{k}\right).\boldsymbol{g}T^2 \\ & + \frac{1}{6} m\boldsymbol{g}^2 T^3 + \hbar \Omega_{ba} y(\boldsymbol{g}_0) T + \left(E_a + \frac{1}{2}\hbar\omega\right) T. \end{aligned} \tag{54}$$

Note that the matrix $U(\lambda)$ introduced in (30) has been fixed by the condition (33), so it has to satisfy the following auxiliary equation:

$$\mathrm{i}\frac{\mathrm{d}U}{\mathrm{d}\lambda}(\lambda) + Q(\lambda)U(\lambda) = 0 \tag{55}$$

i.e. a system of two coupled equations

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}\lambda} + \dfrac{\lambda}{2} A - \mathrm{i}\sqrt{\nu} B = 0, \\ \dfrac{\mathrm{d}B}{\mathrm{d}\lambda} - \dfrac{\lambda}{2} B - \mathrm{i}\sqrt{\nu} A = 0, \end{cases} \tag{56}$$

with

$$\mathrm{A}(\lambda_0) = 1, \quad B(\lambda_0) = 0,$$

and whose solution determines the elements A and B of the matrix $U(\lambda)$.

Let us uncouple this system. Pose

$$\tilde{A}(\lambda) = -\mathrm{e}^{\mathrm{i}\pi/4} \frac{\Omega_{ba}}{\sqrt{\boldsymbol{k}\boldsymbol{g}}} A(\lambda),$$

and the preceding system becomes

$$\begin{cases} \dfrac{\mathrm{d}^2 \tilde{A}}{\mathrm{d}\lambda^2} + \left(-\nu + \dfrac{1}{2} - \dfrac{\lambda^2}{4}\right) \tilde{A} = 0, \\ \dfrac{\mathrm{d}^2 B}{\mathrm{d}\lambda^2} + \left(-\nu - \dfrac{1}{2} - \dfrac{\lambda^2}{4}\right) B = 0, \end{cases} \tag{57}$$

where the boundary conditions are now

$$\tilde{A}(\lambda_0) = -\mathrm{e}^{\mathrm{i}\pi/4} \frac{\Omega_{ba}}{\sqrt{\boldsymbol{k}\boldsymbol{g}}}, \quad B(\lambda_0) = 0. \tag{58}$$

This system allows for a solution of parabolic cylinder function type [7]:

$$\begin{cases} B = \frac{1}{c}\left[D_\nu(-\mathrm{i}\lambda)D_{-\nu-1}(\lambda_0)\right. \\ \quad \left.-D_{-\nu-1}(\lambda)D_\nu(-\mathrm{i}\lambda_0)\right]\mathrm{e}^{\mathrm{i}\pi/4}\frac{\Omega_{ba}}{\sqrt{\boldsymbol{kg}}}, \\ A = \frac{1}{c}\left[D_\nu(-\mathrm{i}\lambda_0)D_{-\nu}(\lambda)\right. \\ \quad \left.+\frac{\Omega_{ba}^2}{\boldsymbol{kg}}D_{-\nu-1}(\lambda_0)D_{\nu-1}(-\mathrm{i}\lambda)\right], \end{cases} \tag{59}$$

with

$$c = D_{-\nu}(\lambda_0)D_\nu(-\mathrm{i}\lambda_0) + \frac{\Omega_{ba}^2}{\boldsymbol{kg}}D_{\nu-1}(-\mathrm{i}\lambda_0)D_{-\nu-1}(\lambda_0). \tag{60}$$

The recurrence relations between the D_ν [7] as well as the boundary conditions are used to determine A and B.

One can see that the result is the same one as that of the [3]. Indeed the state of the atom in the time T can be known starting from its state at time $t = 0$, by using the fundamental equation which uses the propagator

$$\Psi(\boldsymbol{r}, T) = \int \boldsymbol{K}(\boldsymbol{r}, \boldsymbol{r}'; T)\Psi(\boldsymbol{r}', 0)\mathrm{d}^3r'. \tag{61}$$

By replacing the expression (47) in (59) and putting

$$\int \mathrm{d}^3r' \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\boldsymbol{kr}'} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{(\mathrm{i}/2)\phi\boldsymbol{\sigma}_z}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_0\boldsymbol{r}'}\Psi(\boldsymbol{r}', 0) = \begin{pmatrix} a(p_0) \\ b(p_0) \end{pmatrix},$$

we obtain

$$\begin{aligned} &\Psi(\boldsymbol{r}, T) \\ &= \frac{1}{(2\pi\hbar)^{3/2}} \int \mathrm{d}^3p_0 \mathrm{e}^{(\mathrm{i}/\hbar)(m\boldsymbol{g}T+\boldsymbol{p}_0)\boldsymbol{r}}\mathrm{e}^{-(\mathrm{i}/\hbar)E_{\mathrm{g}}(T)} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{kr}} & 0 \\ 0 & 1 \end{pmatrix} \\ &\times \mathrm{e}^{-(\mathrm{i}/2)(\omega T+\phi)\sigma_z} \begin{pmatrix} A(T) & -B^*(T) \\ B(T) & A^*(T) \end{pmatrix} \begin{pmatrix} a(p_0) \\ b(p_0) \end{pmatrix}, \end{aligned} \tag{62}$$

the same wave function as obtained by solution of the Schrödinger equation [3].

The same physical features (probabilities of Rabi, transfer of population,...) can thus be obtained again and the influence of gravitation can be discussed as has been done by [3].

4 Conclusion

We have given an exact treatment using the path-integral formalism to the problem of the two-level atom in interaction with an electromagnetic wave and submitted to gravitation. Thanks to the two fermonic oscillators replacing the spin, the propagator has been written, first in the conventional form $\int \mathcal{D}(\text{path})\exp(\mathrm{i}/\hbar)S(\text{path})$, then determined with exactitude. Thus the wave function has been deduced via an auxiliary equation which admits a solution of a parabolic cylindric type. Our results through the path-integral approach are in accordance with those in [3]

Finally let us note in the passing that the expressions (38) and (39) can be compared to $\sum_\alpha \exp(\mathrm{i}S_\alpha)$ where the sum relates to all the traditional classical paths α (parameterized here by the momentum). This form, remarkable for this problem, shows that a traditional semi-treatment can lead to an exact result. The calculation is currently available. It can be found elsewhere.

References

1. C. Grosche, F. Steiner, Handbook of Feynman path integrals (Springer-Verlag 1998)
2. T. Boudjedaa, A. Bounames, L. Chetouani, T.F. Hammann, J. Math. Phys. **36**, 1602 (1995); Kh. Nouicer, L. Chetouani, Phys. Lett. A **281**, 218 (2001); A. Merdaci, T. Boudjedaa, L. Chetouani, Phys. Scrip. **64**, 19 (2001); Czech. J. Phys. **51**, 865 (2001)
3. Claus Lämmerzahl, Christian J. Bordé, Phys. Lett. A **203**, 59 (1995)
4. A.M. Perelomov, Commun. Math. Phys. **26**, 22 (1972)
5. J. Klauder, Ann. Phys. (NY) **11**, 123 (1960); Coherent states, edited by J.R. Klauder, B. Skagerstam (Word Scientific, Singapore 1985); T. Boudjedaa, A. Bounames, Kh. Nouicer, L. Chetouani, T.F. Hammann, Phys. Scrip. **54**, 225 (1996)
6. Y. Ohnuki, T. Kashiwa, Prog. Theor. Phys. **60**, 548 (1978)
7. I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series, and product (Academic Press, New York 1983)
8. Masato Nakamura, Kazuo Kitahara, J. Phys. Soc. Japan **60**, 1388 (1991)